> *Definition:* We define the nullity of A to be
>
> $$\text{nullity}(A) = \dim(\text{null}(A)) \tag{5}$$

Note 6: If the $m \times n$ matrices A and C are row equivalent, then

$$\text{null}(A) = \text{null}(C) \tag{6}$$

$A\vec{x} = \vec{0}$

$[A \mid \vec{0}] \sim [C \mid \vec{0}]$

$A\vec{x} = \vec{0}$ $\quad$ $C\vec{x} = \vec{0}$

since elementary row operations preserve solutions to $A\mathbf{x} = \mathbf{0}$.

Example 6: Let $A = \begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix}$.

1. Find a basis for null(A). Calculate rank(A) and nullity(A).

$A \underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & 2 \end{bmatrix} \underset{\substack{R_3 := R_3 - 2R_1 \\ R_2 := R_2 + R_1}}{\sim} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{bmatrix} \underset{\substack{R_2 := \frac{1}{2}R_2 \\ R_3 := R_3 + R_2}}{\sim} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underset{R_1 := R_1 - R_2}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$

null(A) = null(C)

$x_1 - 2x_2 + x_4 = 0$

$x_3 + 3x_4 = 0$

Free $\quad x_2 = t_1$

$x_4 = t_2$

$x_1 - 2t_1 + t_2 = 0$

$x_3 + 3t_2 = 0$

$x_1 = 2t_1 - t_2$

$x_3 = -3t_2$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t_1 - t_2 \\ t_1 \\ -3t_2 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

null (A) = null (C) = span $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right)$

rank(A) + null(A) = 4
= n # of col of A,

$\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ Basis for null (A)

nullity $(A) = 2$

rank$(A) = 2$

2. Calculate rank(A) + nullity(A). What do you observe?

Example 7: Let $A = \begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 2 & 0 & -3 & -1 \end{bmatrix}$. $\underset{\sim}{\text{exercise}}$ $\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C$

1. Find a basis for null(A). Calculate rank(A) and nullity(A).

null (A) = null (C).

$x_1 - x_4 + x_5 = 0$

$x_2 - x_4 - x_5 = 0$

$x_3 + x_4 + x_5 = 0$

Fvec

$x_4 = t_1$

$x_5 = t_2$

$x_1 = t_1 - t_2$

$x_2 = t_1 + t_2$

$x_3 = -t_1 - t_2$

$\vec{x}$ in null(A) $\Longleftrightarrow$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 \\ t_1 + t_2 \\ -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = \underbrace{t_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$

$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for null(A).

nullity$(A) = 2$

rank$(A) = 3$

2. Calculate rank(A) + nullity(A). What do you observe?

rank(A) + nullity(A) = $2 + 3 = 5$ = # of columns of A